

MIMO Multichannel Beamforming: SER and Outage Using New Eigenvalue Distributions of Complex Noncentral Wishart Matrices

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Abstract

This paper analyzes MIMO systems with multichannel beamforming in Ricean fading. Our results apply to a wide class of multichannel systems which transmit on the eigenmodes of the MIMO channel. We first present new closed-form expressions for the marginal ordered eigenvalue distributions of complex noncentral Wishart matrices. These are used to characterize the statistics of the signal to noise ratio (SNR) on each eigenmode. Based on this, we present exact symbol error rate (SER) expressions. We also derive closed-form expressions for the diversity order, array gain, and outage probability. We show that the global SER performance is dominated by the subchannel corresponding to the minimum channel singular value. We also show that, at low outage levels, the outage probability varies inversely with the Ricean K -factor for cases where transmission is only on the most dominant subchannel (i.e. a singlechannel beamforming system). Numerical results are presented to validate the theoretical analysis.

Index Terms

Multiple-input multiple-output systems, Ricean fading, Wishart matrices, multichannel beamforming.

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I. INTRODUCTION

Multiple-input multiple-output (MIMO) communication systems have received considerable attention in recent years as they offer substantial capacity improvements over conventional single-input single-output (SISO) systems [1, 2], with no penalty in either power or bandwidth. When perfect channel state information (CSI) is available at both the transmitter and receiver, it is well known [1] that the capacity-achieving strategy is to transmit on the eigenmodes of the MIMO channel using linear transmit-receive processing (hereafter referred to as multichannel beamforming (MB)), upon which independent Gaussian codes with water-filling power allocation are employed. In practice, the high complexity requirements of Gaussian-like codes are often prohibitive, and either suboptimally-coded or uncoded transmission is used. Interestingly, for these suboptimal systems, it has been shown that under most performance criteria of practical interest (e.g. symbol error rate (SER), mean-square error (MSE), among others), MB still corresponds to the optimal choice of linear transmit-receive processing (in some cases up to a rotation matrix) [3].

In this paper we provide an analytical investigation of the performance of uncoded MB MIMO systems in Ricean fading channels. Although these systems are particularly appealing from a practical point of view, there are currently very few related analytical performance results available in the literature. In [4], the global SER (i.e. SER averaged over all subchannels) was derived for Rayleigh and Ricean fading channels under the assumption that all available subchannels were used for transmission, with equal power, and with the same (BPSK) modulation. It was shown in [5] however (via Monte Carlo simulations), that the performance is significantly improved by transmitting on only a *subset* of the available subchannels, and also by using different powers and constellations on each of these. The global SER results of [4] cannot be easily generalized to these important scenarios. In [6], global and per-subchannel SER expressions were presented for MB MIMO systems in Rayleigh fading in the high signal to noise ratio (SNR) regime, allowing for transmission on a set of arbitrarily selected subchannels. In this paper we consider the more general class of Ricean fading channels, and seek performance measures for all SNRs.

The main difficulty in obtaining analytical performance results for MB MIMO systems in Ricean fading is that it requires the marginal statistical distributions of the ordered eigenvalues of complex noncentral Wishart matrices. Although many results are available for the eigenvalue statistics of complex *central* Wishart matrices (see [7–13], and references therein), there are very few results

for the noncentral case. In [14], the joint probability density function (p.d.f.) of these ordered eigenvalues was obtained for the special case of full-rank non-centrality matrices. This joint p.d.f. was used in [15] to derive the exact p.d.f. and cumulative distribution function (c.d.f.) of the dual-antenna MIMO capacity in Ricean channels with rank-1 mean matrices. Unfortunately the joint p.d.f. involves a hypergeometric function of matrix arguments and Vandermonde determinants, and is not easily marginalized. In [16], the p.d.f. and c.d.f. for the *maximum* eigenvalue was obtained for the special cases of rank-1 and full-rank non-centrality matrices. These results were used to analyze the statistics of the output SNR of MIMO maximum-ratio combining (MIMO-MRC) systems in Ricean channels.

In this paper we derive new exact closed-form expressions for the marginal distributions of *all* of the ordered eigenvalues of complex noncentral Wishart matrices. The results apply for non-centrality matrices of *arbitrary* rank. Explicit expressions are given for the marginal c.d.f.s, from which the closed-form marginal p.d.f.s can be obtained trivially via differentiation. These marginal c.d.f.s are used to analyze the performance of MB MIMO systems in Ricean channels with mean matrices of arbitrary rank. New exact expressions are presented for both the subchannel SERs and the global SER. These expressions are general in the sense that they allow for transmission on any arbitrary number of subchannels, with possibly unequal signal constellations, and with possibly unequal powers¹. The exact c.d.f. expressions are also used to obtain new exact closed-form expressions for the outage probability of MB MIMO systems.

In addition to the exact marginal distributions, we also derive new first-order asymptotic expansions for the marginal c.d.f.s and p.d.f.s of the ordered eigenvalues of complex noncentral Wishart matrices. These expansions are particularly useful for gaining further insights into the effect of various system parameters on the performance of MB MIMO systems. In particular, the asymptotic p.d.f. expansions are used to derive explicit expressions for the diversity order and array gain, which are the factors governing the SER performance at high SNR. These expressions reveal that the global SER is dominated by the subchannel corresponding to the minimum channel singular value. The asymptotic c.d.f. expansions are used to examine the outage probability of MB MIMO at low (practical) outage levels, when only a single subchannel is selected for transmission (corresponding to MIMO-MRC transmission). We find that in this outage regime, the outage probability becomes independent of the rank of the channel mean, and varies inversely with the

¹Note that while we allow for unequal powers, we assume that they remain fixed; and we calculate the average SER. In other words, we are not solving the waterfilling problem, for which the power levels become functions of the eigenvalues and change at the fading rate of the channel.

Ricean K -factor.

II. SYSTEM MODEL

Consider a MIMO system with m transmit and n receive antennas, modeled as

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}, \quad (1)$$

where $\mathbf{y} \in \mathbb{C}^{n \times 1}$ is the discrete-time received vector, $\mathbf{s} \in \mathbb{C}^{m \times 1}$ is the transmitted vector, and $\mathbf{n} \in \mathbb{C}^{n \times 1}$ is the noise vector with i.i.d. entries $\sim \mathcal{CN}(0, 1)$. Also, $\mathbf{H} \in \mathbb{C}^{n \times m}$ is the Ricean fading channel matrix, which is decomposed as follows [17]

$$\mathbf{H} = \varepsilon\sqrt{K}\bar{\mathbf{H}} + \varepsilon\tilde{\mathbf{H}}, \quad (2)$$

where $\bar{\mathbf{H}}$ is the (arbitrary) rank- L deterministic channel component satisfying $\text{tr}(\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger) = mn$, and $\tilde{\mathbf{H}}$ is the random (scattered) channel component containing i.i.d. $\mathcal{CN}(0, 1)$ entries. The parameter K is the Ricean K -factor, which is the ratio between the energy in $\bar{\mathbf{H}}$ and the average energy in $\tilde{\mathbf{H}}$, and $\varepsilon = 1/\sqrt{K+1}$ is a power normalization constant. Note that \mathbf{H} in (2) follows a complex matrix-variate Gaussian distribution with mean matrix $\mathbf{M} = \sqrt{K}\varepsilon\bar{\mathbf{H}}$ and (column) correlation matrix $\varepsilon^2\mathbf{I}_n$. Adopting standard notation from multivariate statistical theory (e.g. see [18, 19]), this distribution is denoted

$$\mathbf{H} \sim \mathcal{CN}_{n,m}(\mathbf{M}, \varepsilon^2\mathbf{I}_n \otimes \mathbf{I}_m). \quad (3)$$

Let us now define

$$\mathbf{W} = \begin{cases} \mathbf{H}\mathbf{H}^\dagger & n \leq m \\ \mathbf{H}^\dagger\mathbf{H} & n > m \end{cases} \quad (4)$$

$s = \min(n, m)$, and $t = \max(n, m)$. With these definitions, $\mathbf{W} \in \mathbb{C}^{s \times s}$ follows a complex noncentral Wishart distribution, denoted

$$\mathbf{W} \sim \mathbf{W}_s(t, \mathbf{\Sigma}, \mathbf{\Omega}), \quad (5)$$

where $\mathbf{\Sigma} = \varepsilon^2\mathbf{I}_s$ and

$$\mathbf{\Omega} = \begin{cases} \mathbf{\Sigma}^{-1}\mathbf{M}\mathbf{M}^\dagger & n \leq m \\ \mathbf{\Sigma}^{-1}\mathbf{M}^\dagger\mathbf{M} & n > m \end{cases} \quad (6)$$

is the arbitrary-rank non-centrality matrix.

We consider the class of MB MIMO spatial multiplexing systems considered in [3, 5, 6]. As in [3, 5, 6], we assume that perfect CSI is known at both the transmitter and receiver. The transmit vector can be written as

$$\mathbf{s} = \mathbf{B}\mathbf{x}, \quad (7)$$

where $\mathbf{B} \in \mathbb{C}^{m \times r}$ is the transmit precoder matrix which maps the $r \leq \min(m, n)$ modulated data symbols x_i (elements of \mathbf{x} , with $E\{\mathbf{x}\mathbf{x}^\dagger\} = \mathbf{I}_r$, and chosen from possibly different signal constellations), onto the m transmit antennas, and is normalized according to

$$E\{\|\mathbf{s}\|^2\} = \text{tr}(\mathbf{B}\mathbf{B}^\dagger) \leq P \quad (8)$$

where P is the average signal to noise ratio (SNR) per receive antenna. The estimated vector at the receiver is given by

$$\hat{\mathbf{x}} = \mathbf{A}^\dagger \mathbf{y}, \quad (9)$$

where $\mathbf{A}^\dagger \in \mathbb{C}^{r \times n}$ is the receive (spatial) equalizer matrix.

It was shown in [3], that under many practical design criteria (such as maximizing the mutual information, minimizing the arithmetic or geometric mean-square error, among others) the optimal transmit and receive filters result in a MB system, and are given by

$$\mathbf{B} = \mathbf{U}_H \mathbf{P} \quad (10)$$

and

$$\mathbf{A} = (\mathbf{H}\mathbf{B}\mathbf{B}^\dagger\mathbf{H}^\dagger + \mathbf{I}_n)^{-1} \mathbf{H}\mathbf{B} \quad (11)$$

respectively, where $\mathbf{U}_H \in \mathbb{C}^{n \times r}$ has as columns the eigenvectors corresponding to the r largest eigenvalues of $\mathbf{H}^\dagger\mathbf{H}$, and $\mathbf{P} = \text{diag}\left(\{\sqrt{p_i}\}_{i=1,\dots,r}\right)$ is a power allocation matrix, with $p_i > 0$ and $\sum_i p_i = P$.

With this choice of linear transmit and receive filtering, the MIMO channel is decomposed into r parallel scalar (eigenmode) subchannels, which are described as follows

$$\hat{x}_k = \kappa_k \left(\varepsilon \sqrt{\phi_k p_k} x_k + n_k \right), \quad k = 1, \dots, r \quad (12)$$

where κ_k is a constant (which does not affect the received subchannel SNR), \hat{x}_k and n_k are the

k th elements of $\hat{\mathbf{x}}$ and \mathbf{n} respectively, and ϕ_k is the k th largest eigenvalue of

$$\mathbf{S} = \mathbf{\Sigma}^{-1} \mathbf{W} \sim \mathbf{W}_s(t, \mathbf{I}_s, \mathbf{\Omega}). \quad (13)$$

The instantaneous received subchannel SNR is given by

$$\gamma_k = \varepsilon^2 \phi_k p_k \quad k = 1, \dots, r \quad (14)$$

Clearly the SNR (and therefore the performance) for each subchannel, as well as the overall received SNR (and global performance), depend explicitly on the marginal statistical distributions of the eigenvalues $\phi_1 > \dots > \phi_r$ of the complex noncentral Wishart matrix in (13). In the following section we will present new closed-form exact and asymptotic expressions for the marginal distributions of these eigenvalues. These results will then be used in Section IV to analyze the performance of MB MIMO systems in Ricean fading channels.

III. NEW STATISTICAL PROPERTIES OF THE ORDERED EIGENVALUES OF COMPLEX NONCENTRAL WISHART MATRICES

A. New Exact Ordered Eigenvalue Distribution Results

In this subsection we derive new *exact* closed-form marginal eigenvalue c.d.f. expressions. Note that exact marginal p.d.f. expressions can also be easily obtained from these c.d.f. results via differentiation. These results, however, are omitted due to space constraints. For convenience, we consider the smallest, largest, and k th largest eigenvalues separately. The proofs of all results in this section are given in the appendices.

First consider the smallest eigenvalue ϕ_s . It should be noted that, in addition to the performance analysis of MB MIMO systems considered in this paper, the statistical properties of the smallest eigenvalue of Wishart matrices are important in the analysis of various other MIMO systems and applications (see e.g. [20–23]).

Theorem 1: The c.d.f. of the smallest eigenvalue ϕ_s of the complex noncentral Wishart matrix \mathbf{S} in (13) is given by

$$F_{\phi_s}(x) = 1 - |\Psi(x)| / |\Psi(0)|, \quad (15)$$

where $\Psi(x)$ is an $s \times s$ matrix function of $x \in (0, \infty)$ whose entries are given by

$$\{\Psi(x)\}_{i,j} = \begin{cases} 2^{(2i-s-t)/2} Q_{s+t-2i+1, t-s}(\sqrt{2\lambda_j}, \sqrt{2x}) & j = 1, \dots, L \\ \Gamma(t+s-i-j+1, x) & j = L+1, \dots, s \end{cases} \quad (16)$$

and where $\lambda_1 > \dots > \lambda_L$ are the L ($\leq s$) non-zero eigenvalues of $\mathbf{\Omega}$. Also, $Q_{p,q}(a, b)$ is the Nuttall Q -function, defined in [24] by

$$Q_{p,q}(a, b) = \int_b^\infty x^p \exp\left(-\frac{x^2 + a^2}{2}\right) I_q(ax) dx, \quad (17)$$

$I_q(\cdot)$ is the q th order modified Bessel function of the first kind, and $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function, defined as [25]

$$\Gamma(p, x) = \int_x^\infty t^{p-1} e^{-t} dt = (p-1)! e^{-x} \sum_{k=0}^{p-1} \frac{x^k}{k!}, \quad p = 1, 2, \dots \quad (18)$$

Proof: See Appendix I. \square

Note that the result in (15) can be easily programmed and efficiently evaluated. Moreover, the sum of the Nuttall Q -function indices in (16) is *odd*. As such, this function has a closed-form representation given in [26, Eq. 8].

Now consider the largest eigenvalue ϕ_1 of \mathbf{S} . In [16], the c.d.f. of ϕ_1 was derived for the particular cases where the mean matrix was either rank-1 or full-rank, and used to analyze the performance of MIMO-MRC systems in Ricean channels. The following theorem presents a new, simpler expression for this c.d.f., and also generalizes the results of [16] as it applies for mean matrices with *arbitrary* rank.

Theorem 2: The c.d.f. of the largest eigenvalue ϕ_1 of the complex noncentral Wishart matrix \mathbf{S} in (13) is given by

$$F_{\phi_1}(x) = |\mathbf{\Xi}(x)| / |\mathbf{\Psi}(0)|, \quad (19)$$

where $\mathbf{\Xi}(x)$ is an $s \times s$ matrix function of $x \in (0, \infty)$ whose entries are given by

$$\{\mathbf{\Xi}(x)\}_{i,j} = \begin{cases} 2^{(2i-s-t)/2} [Q_{s+t-2i+1,t-s}(\sqrt{2\lambda_j}, 0) - Q_{s+t-2i+1,t-s}(\sqrt{2\lambda_j}, \sqrt{2x})] & j = 1, \dots, L \\ \gamma(t+s-i-j+1, x) & j = L+1, \dots, s \end{cases} \quad (20)$$

where $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function, given by

$$\gamma(p, x) = \int_0^x t^{p-1} e^{-t} dt = (p-1)! \left(1 - e^{-x} \sum_{k=0}^{p-1} \frac{x^k}{k!}\right), \quad p = 1, 2, \dots \quad (21)$$

Proof: See Appendix II. \square

Finally we consider the k th largest eigenvalue ϕ_k of \mathbf{S} .

Theorem 3: The c.d.f. of the k th largest eigenvalue ϕ_k of the complex noncentral Wishart matrix

\mathbf{S} in (13), where $k > 1$, is given by

$$F_{\phi_k}(x) = F_{\phi_{k-1}}(x) + \Pr(\phi_s < \dots < x < \phi_{k-1} < \dots < \phi_1) = F_{\phi_{k-1}}(x) + p, \quad (22)$$

where

$$p = c_3 \sum_1 |\Theta(x)|, \quad (23)$$

$$c_3 = \frac{\prod_{i=1}^L \left(\lambda_i^{(2L-s-t)/2} \right)}{\Gamma_{s-L}(s-L) \prod_{i < j}^L (\lambda_i - \lambda_j)}, \quad (24)$$

$$\{\Theta(x)\}_{\alpha_{i,j}} = \begin{cases} \{\Psi(x)\}_{\alpha_{i,j}} & i = 1, \dots, k-1 \\ \{\Xi(x)\}_{\alpha_{i,j}} & i = k, \dots, s \end{cases} \quad (25)$$

\sum_1 indicates the summation over the combination $(\alpha_1 < \alpha_2 < \dots < \alpha_{k-1})$ and $(\alpha_k < \alpha_{k+1} < \dots < \alpha_s)$, $(\alpha_1, \dots, \alpha_s)$ being a permutation of $(1, \dots, s)$.

Proof: See Appendix III. □

Remark: Let $\omega_1 \geq \omega_2 \geq \dots \geq \omega_s \geq 0$ be the ordered singular values of \mathbf{H} . Recalling that $\mathbf{S} = \Sigma^{-1}\mathbf{W}$, we have the following relationship

$$F_{\omega_s}(x) = F_{\phi_s}(\varepsilon^{-2}x^2). \quad (26)$$

B. New Asymptotic Ordered Eigenvalue Distribution Results

In this subsection we present new asymptotic *first-order expansions* of the marginal eigenvalue p.d.f.s and c.d.f.s. of complex noncentral Wishart matrices. These will be particularly useful for deriving the diversity order and array gain of MB MIMO systems in the following section, as well as for analyzing the asymptotic outage probability. Note that a first order expansion of the p.d.f. of the k th largest eigenvalue of complex *central* Wishart matrices was presented in [6].

Theorem 4: The first order expansions of the marginal p.d.f. and c.d.f. of the k th largest eigenvalue ϕ_k of the complex noncentral Wishart matrix \mathbf{S} in (13), where $1 \leq k \leq s$, are given respectively by

$$f_{\phi_k}(\phi_k) = a_k \phi_k^{d_k} + o(\phi_k^{d_k}) \quad (27)$$

and

$$F_{\phi_k}(\phi_k) = \frac{a_k}{d_k + 1} \phi_k^{d_k+1} + o(\phi_k^{d_k+1}), \quad (28)$$

where

$$d_k = s_k t_k - 1 \quad (29)$$

and

$$s_k = s - k + 1, \quad t_k = t - k + 1. \quad (30)$$

Also, a_k is given for $k = 1$ by

$$a_1 = \frac{st \Gamma_s(s)}{\Gamma_s(t+s)} e^{-\text{tr}(\Omega)} \quad (31)$$

and for $k > 1$ by

$$a_k = \frac{s_k t_k \Gamma_{k-1}(s) \Gamma_{s_k}(s_k)}{\Gamma_{s_k}(t_k + s_k)} c_3 \left(\prod_{i=1}^L \lambda_i^{(s-t)/2} \right) |\mathbf{X}| \quad (32)$$

where $\Gamma_s(\cdot)$ is the normalized complex multivariate gamma function, defined in (48), and \mathbf{X} is an $s \times s$ matrix with (i, j) th entry

$$\{\mathbf{X}\}_{i,j} = \begin{cases} L_{s-i}^{(t-s)}(-\lambda_j) & i = 1, \dots, k-1, \quad j = 1, \dots, L \\ \lambda_j^{s-i} e^{-\lambda_j} & i = k, \dots, s, \quad j = 1, \dots, L \\ \binom{t-i}{j-i} & i = 1, \dots, k-1, \quad j = L+1, \dots, s, \quad j \geq i \\ 0 & i = 1, \dots, k-1, \quad j = L+1, \dots, s, \quad j < i \\ \frac{(-1)^{i-j} (s-j)!}{(i-j)!} & i = k, \dots, s, \quad j = L+1, \dots, s, \quad j \leq i \\ 0 & i = k, \dots, s, \quad j = L+1, \dots, s, \quad j > i \end{cases} \quad (33)$$

where

$$L_k^{(n)}(x) = \sum_{i=0}^k \binom{k+n}{k-i} \frac{(-x)^i}{i!} \quad (34)$$

is the generalized k th-order Laguerre polynomial.

Proof: See Appendix IV. □

IV. PERFORMANCE ANALYSIS OF MIMO MULTICHANNEL BEAMFORMING IN RICEAN FADING CHANNELS

A. Symbol Error Rate Analysis

We now analyze the SER performance of the MB MIMO systems introduced in Section II. For many general modulation formats (see below), the average SER of the k th subchannel can be

expressed as [27]

$$\text{SER}_k = E_{\gamma_k} \left\{ \alpha_k Q \left(\sqrt{2\beta_k \gamma_k} \right) \right\}, \quad k = 1, \dots, r \quad (35)$$

where $Q(\cdot)$ is the Gaussian Q-function, and α_k and β_k are modulation-specific constants. Some example modulation formats for which (35) apply include BPSK ($\alpha_k = 1, \beta_k = 1$); BFSK with orthogonal signalling ($\alpha_k = 1, \beta_k = 0.5$) or minimum correlation ($\alpha_k = 1, \beta_k = 0.715$); and M -ary PAM ($\alpha_k = 2(M-1)/M, \beta_k = 3/(M^2-1)$). Our results also provide the approximate SER for those other formats for which (35) is an approximation, e.g. M -ary PSK ($\alpha_k = 2, \beta_k = \sin^2(\pi/M)$) [27, Eq. 5.2-61]. Using results from [10] and [28], (35) can be expressed in the following equivalent form

$$\text{SER}_k = \frac{\alpha_k \sqrt{\beta_k}}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\beta_k u}}{\sqrt{u}} F_{\phi_k} \left(\frac{u}{\varepsilon^2 p_k} \right) du, \quad k = 1, \dots, r \quad (36)$$

where we have used the fact that $F_{\gamma_k}(u) = F_{\phi_k}(\varepsilon^{-2}u/p_k)$. Hence, by applying *Theorems 1-3* in (36) we obtain an exact expression for the average SER of each subchannel. Although it does not appear that the integrals in (36) can be evaluated in closed form, numerical integration can be performed to evaluate SER_k much more efficiently than is possible via Monte Carlo simulation.

The global SER can be derived from the subchannel SERs as follows; since independent symbols are sent on each subchannel during each channel use:

$$\text{SER} = \frac{1}{r} \sum_{k=1}^r \text{SER}_k. \quad (37)$$

To gain further insights, we now consider the SER at high SNR. We will restrict this asymptotic analysis to systems with uniform power allocation (i.e. $p_k = P/r$), since (as mentioned in [6]) most of the practical power allocation solutions in [3] tend to uniform as the power is increased. In the high SNR regime, the key factors governing system performance are the diversity order and array gain. We will now present closed-form expressions for these factors.

Using a general SISO SER result from [29], we find that in our case SER_k can be approximated in the high SNR regime by considering a first order expansion of the p.d.f. of ϕ_k as $\phi_k \rightarrow 0^+$. Hence, using the result from [29], along with *Theorem 4*, we obtain a high SNR subchannel SER expression given by

$$\text{SER}_k^\infty = (G_a(k) \cdot P)^{-G_d(k)} + o(P^{-G_d(k)}), \quad k = 1, \dots, r \quad (38)$$

where the diversity order is

$$G_d(k) = d_k + 1 \quad (39)$$

and the array gain is

$$G_a(k) = \frac{2\beta_k \varepsilon^2}{r} \left(\frac{\alpha_k 2^{d_k} a_k \Gamma(d_k + 3/2)}{\sqrt{\pi} (d_k + 1)} \right)^{-1/(d_k+1)}. \quad (40)$$

Comparing (39) with the i.i.d. Rayleigh results presented previously in [6], we see that the subchannel diversity orders are the same in both Rayleigh and Ricean channels. Moreover, since $G_d(1) \geq G_d(2) \geq \dots \geq G_d(r)$, the r th subchannel has the poorest performance in terms of average SER. Using (37), the global average SER of MB MIMO systems at high SNR can be obtained as

$$\text{SER}^\infty = \frac{1}{r} (G_a(r) \cdot P)^{-G_d(r)} + o(P^{-G_d(r)}) \quad (41)$$

which is clearly dominated by the r th subchannel SER (i.e. the subchannel corresponding to the smallest singular value).

B. Outage Probability Analysis

We now consider the outage probability of MB MIMO systems in Ricean fading channels. The outage probability is an important quality of service measure, defined as the probability that the received SNR drops below an acceptable SNR threshold γ_{th} . For convenience, we assume an equal power allocation strategy is employed. In this case, the subchannel SNRs are ordered according to $\gamma_1 > \dots > \gamma_r$ in (14), and the outage probability of the overall MB MIMO system is dominated by the subchannel corresponding to $k = r$. As such, the outage probability is obtained exactly from *Theorems 1-3* as follows

$$P_{\text{out}} = \Pr(\gamma_r \leq \gamma_{\text{th}}) = F_{\phi_r} \left(\frac{\gamma_{\text{th}} (K+1) r}{P} \right). \quad (42)$$

We note that for the special case $r = 1$ (i.e. only the best subchannel corresponding to ϕ_1 is used), the MB MIMO system we consider is equivalent to the MIMO-MRC systems considered in [11, 16]. For these systems, outage probability expressions were derived previously in [16] for the special case of channels with rank-1 and full-rank mean matrices. Our result (42) is clearly more general as it applies for all $r \geq 1$ and for mean matrices of arbitrary rank. Moreover, for the special case $r = 1$, our result is simpler than the previous results given in [16].

In practice, we are usually interested in small outage probabilities (i.e. 0.01, 0.001, ...), which correspond to small values of γ_{th} . To gain further intuition at these small outage probabilities, let us consider the case $r = 1$ (i.e. the MIMO-MRC case), and use (28) in *Theorem 4* to write the outage probability in (42) as follows

$$\tilde{P}_{\text{out}} = \frac{\Gamma_s(s) e^{-\text{tr}(\Lambda)}}{\Gamma_s(t+s)} \left(\frac{\gamma_{\text{th}}(K+1)}{P} \right)^{st} + o((\gamma_{\text{th}})^{st}). \quad (43)$$

Since $\text{tr}(\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger) = st$ (see Section II) we can further simplify to obtain

$$\tilde{P}_{\text{out}} = \frac{\Gamma_s(s)}{\Gamma_s(t+s)} \left(\frac{\gamma_{\text{th}}}{P} \right)^{st} \frac{(K+1)^{st}}{e^{Kst}} + o((\gamma_{\text{th}})^{st}). \quad (44)$$

This explicitly shows that for MIMO-MRC transmission the outage probability (at low outage levels) does not depend on the rank of the channel mean. Moreover, since

$$\frac{d}{dK} (K+1)^{st} e^{-Kst} = -Kst e^{-Kst} (K+1)^{st-1} < 0, \quad K > 0 \quad (45)$$

we see that the outage probability varies inversely with the Ricean K -factor.

V. NUMERICAL RESULTS

For our numerical results we consider a 3×5 Ricean MIMO channel and, unless otherwise specified, a rank-3 deterministic component $\bar{\mathbf{H}}$ with singular values $\{2.9751, 2.2840, 0.9657\}$, which were randomly generated to verify the analysis.

It is important to note, however, that all of the analytic results in this paper apply for arbitrary antenna configurations, and for arbitrary deterministic channel components.

Fig. 1 gives the marginal c.d.f.s of the ordered channel singular values. The analytical curves are generated using *Theorems 1-3* and the singular value relationship (26), and the simulated curves are generated based on 100,000 channel realizations. The figure shows perfect agreement between the analytical results and the simulations.

Fig. 2 shows the c.d.f. of the smallest singular value for different Ricean K -factors, and for rank-1 and rank-3 mean matrices. The deterministic component for the rank-1 case was generated based on the channel model in [15]. As expected, for both mean matrices, as K becomes small (i.e. $K = -10\text{dB}$), the c.d.f.s converge to that of the smallest singular value of a Rayleigh channel.

Fig. 3 shows the exact subchannel SER curves based on (36), exact global SER curve based on (37), and Monte-Carlo SER simulation results, for a MB MIMO system with $K = 0\text{dB}$. All subchannels are used with BPSK modulation ($\alpha_k = 1$, $\beta_k = 1$) and uniform power allocation.

High SNR curves based on (38) and (41) are also presented. In all cases, there is exact agreement between the analytical SER results and the Monte-Carlo simulations, and the diversity order and array gains predicted by the high SNR analytical results are accurate. We also see that the SERs of the 1st and 2nd subchannels are *significantly* better than the 3rd subchannel SER. This suggests that further performance improvements may be possible by using only a subset of the subchannels for transmission (with higher order constellations). The following figure investigates this further.

Fig. 4 shows the analytical and Monte-Carlo simulated global SER curves for MB MIMO systems with different numbers of r active subchannels. The analytical curves are generated based on (37). Uniform power allocation is assumed and, for a fair comparison, the overall rate is set to 3 bits/s/Hz in each case. For $r = 1$, 8PSK ($\alpha_1 = 2$, $\beta_1 = 0.146$) is employed; for $r = 2$ we use QPSK ($\alpha_1 = 2$, $\beta_1 = 0.5$) for the first subchannel and BPSK for the second subchannel; and for $r = 3$ we use BPSK for all subchannels. For the $r = 3$ case see an exact agreement between the analytical and Monte-Carlo simulated curves. As discussed in Section IV-A, (37) only provides an approximation for QPSK and 8PSK, however we see from the $r = 2$ and $r = 3$ curves that the approximation is accurate. In particular, for low to moderate SERs (i.e. $\text{SER} < 10^{-3}$) the analytical curves match almost exactly with the simulated curves. We also see that the SER for the $r = 1$ and $r = 2$ cases is significantly better than for the case where all subchannels are used, which is in agreement with previous Rayleigh fading observations in [5] (via Monte-Carlo simulations). Moreover, we see that the $r = 1$ system has a higher diversity order than the $r = 2$ system (since we've shown the diversity order to be dominated by lowest singular value subchannel), but is shifted to the right due to the higher order constellations. This motivates the design of practical adaptive subchannel selection algorithms, which is an interesting topic for future work, but beyond the scope of this paper.

Fig. 5 shows analytical and Monte-Carlo simulated outage probability curves for a MB MIMO systems with $r = 1$ (i.e. MIMO-MRC transmission), comparing different Ricean K -factors. The analytical curves are generated based on (42). We see an exact agreement between the analytical and simulated curves in all cases. We also see that increasing the Ricean K -factor results in a reduction in outage probability (and an improvement in system performance) at low outage levels. This agrees with the analytic conclusions given in Section IV-B. It is also interesting to observe that the opposite occurs in the high outage regime (i.e. increasing the K -factor increases the outage probability).

VI. CONCLUSION

We have examined the performance of MIMO systems employing multichannel beamforming in arbitrary-rank Ricean channels. Our results are based on new closed-form exact and asymptotic expressions which we have derived for the marginal ordered eigenvalue distributions of complex noncentral Wishart matrices. We have presented exact and high-SNR SER expressions, and derived the diversity order and array gain. Our results have shown that the global SER performance is dominated by the subchannel SER corresponding to the minimum channel singular value. We have also derived new closed-form exact expressions for the outage probability and, for the case of MIMO-MRC transmission, have shown that for outage levels of practical interest, the outage probability varies inversely with the Ricean K -factor.

APPENDIX I

PROOF OF THEOREM 1

We require the following Lemma, which gives the joint eigenvalue p.d.f. of \mathbf{S} for the case where the non-centrality matrix $\mathbf{\Omega}$ has *arbitrary-rank*.

Lemma 1: The joint p.d.f. of the ordered eigenvalues $\phi_1 > \phi_2 > \dots > \phi_s > 0$ of the complex noncentral Wishart matrix \mathbf{S} in (13) is given by

$$f(\phi_1, \dots, \phi_s) = c_1 |\Upsilon| \prod_{i < j}^s (\phi_i - \phi_j) \prod_{k=1}^s \phi_k^{t-s} e^{-\phi_k}, \quad (46)$$

where

$$c_1 = \frac{e^{-\text{tr}(\mathbf{\Omega})} ((t-s)!)^{-s}}{\Gamma_{s-L}(s-L) \prod_{i=1}^L (\lambda_i^{s-L}) \prod_{i < j}^L (\lambda_i - \lambda_j)} \quad (47)$$

and where

$$\Gamma_s(t) = \prod_{i=1}^s (t-i)! . \quad (48)$$

Also Υ is an $s \times s$ matrix with (i,j) th entry

$$\{\Upsilon\}_{i,j} = \begin{cases} {}_0F_1(t-s+1; \lambda_j \phi_i) & i = 1, \dots, s, \quad j = 1, \dots, L \\ \frac{\phi_i^{s-j} (t-s)!}{(t-j)!} & i = 1, \dots, s, \quad j = L+1, \dots, s \end{cases} \quad (49)$$

where ${}_0F_1(\cdot)$ is the scalar Bessel-type hypergeometric function.

Proof: We start by combining a result from [14], which gave the joint eigenvalue p.d.f. for the special case of *full-rank* $\mathbf{\Omega}$, with a hypergeometric function determinant result from [30], to

express the joint eigenvalue p.d.f. in the full-rank Ω case as follows

$$f_{\text{FR}}(\phi_1, \dots, \phi_s) = \frac{e^{-\text{tr}(\Omega)} e^{-\text{tr}(\Phi)}}{((t-s)!)^s} \prod_{i < j}^s (\phi_i - \phi_j) \prod_{k=1}^s \phi_k^{t-s} \frac{|{}_0F_1(t-s+1; \lambda_j \phi_i)|}{\prod_{i < j}^s (\lambda_i - \lambda_j)} \quad (50)$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_s)$, $\lambda_1 > \dots > \lambda_s > 0$ are the eigenvalues of Ω , and $\Phi = \text{diag}(\phi_1, \dots, \phi_s)$.

We generalize this result to arbitrary-rank matrices Ω by taking limits

$$f(\phi_1, \dots, \phi_s) = \lim_{\lambda_{L+1}, \dots, \lambda_s \rightarrow 0} f_{\text{FR}}(\phi_1, \dots, \phi_s) = \frac{e^{-\text{tr}(\Omega)} e^{-\text{tr}(\Phi)}}{((t-s)!)^s} \prod_{i < j}^s (\phi_i - \phi_j) \prod_{k=1}^s \phi_k^{t-s} \mathcal{L} \quad (51)$$

where

$$\mathcal{L} = \lim_{\lambda_{L+1}, \dots, \lambda_s \rightarrow 0} \frac{|f_i(\lambda_j)|}{\prod_{i < j}^s (\lambda_i - \lambda_j)} \quad (52)$$

with $f_i(\lambda_j) = {}_0F_1(t-s+1; \lambda_j \phi_i)$. To evaluate these limits we apply [31, Lemma 2] to obtain

$$\lim_{\lambda_{L+1}, \dots, \lambda_s \rightarrow 0} \frac{|f_i(\lambda_j)|}{\prod_{i < j}^s (\lambda_i - \lambda_j)} = \frac{\begin{vmatrix} f_1(\lambda_1) & \dots & f_1(\lambda_L) & f_1^{(s-L-1)}(0) & \dots & f_1^{(0)}(0) \\ \vdots & \vdots & & & \vdots & \vdots \\ f_s(\lambda_1) & \dots & f_s(\lambda_L) & f_s^{(s-L-1)}(0) & \dots & f_s^{(0)}(0) \end{vmatrix}}{\prod_{i < j}^L (\lambda_i - \lambda_j) \left(\prod_{i=1}^L \lambda_i^{s-L} \right) \Gamma_{s-L}(s-L)} \quad (53)$$

where the required derivatives are easily evaluated as

$$f_i^{(k)}(0) = \frac{\phi_i^k (t-s)!}{(t-s+k)!}. \quad (54)$$

The result now follows by substituting (52)-(54) into (51) and simplifying. \square

We now proceed to evaluate the c.d.f. of the minimum eigenvalue ϕ_s as follows

$$F_{\phi_s}(x) = 1 - \Pr(\phi_1 > \dots \phi_s > x) = 1 - \int_{D_1} f(\phi_1, \dots, \phi_s) d\phi_1 \dots d\phi_s \quad (55)$$

where $D_1 = \{x < \phi_s < \dots < \phi_1\}$. To evaluate the integrals in (55) we require the following result

$$|\Upsilon| \prod_{i < j}^s (\phi_i - \phi_j) = |\Upsilon| |\phi_i^{s-j}| = \sum_{\sigma} \sum_{\mu} \text{sgn}(\mu) \prod_{k=1}^s \phi_{\sigma_k}^{s-k} \{\Upsilon\}_{\sigma_k, \mu_k}, \quad (56)$$

which is easily obtained using *Lemma 1* and the definition of the determinant. In (56), $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_s)$ are permutations of $(1, \dots, s)$, the sums are over all permutations, and $\text{sgn}(\cdot)$ denotes the permutation sign. Substituting (46) and (56) into (55), and

using [12, Lemma 1], it can be shown that the c.d.f. of ϕ_s can be written as

$$F_{\phi_s}(x) = 1 - \Pr(\phi_s > x) = 1 - c_1 \left| \int_x^{+\infty} f_i(\lambda_j, y) dy \right|, \quad (57)$$

where

$$f_i(\lambda_j, y) = \begin{cases} y^{t-i} e^{-y} {}_0F_1(t-s+1; \lambda_j y) & j = 1, \dots, L \\ y^{t+s-i-j} e^{-y} (t-s)! / (t-j)! & j = L+1, \dots, s \end{cases} \quad (58)$$

Using the relation

$${}_0F_1(t-s+1; x) = (t-s)! x^{-(t-s)/2} I_{t-s}(2\sqrt{x}) \quad (59)$$

as well as (17), the remaining integral in (57) can be evaluated as follows

$$\begin{aligned} \mathcal{I}_{i,j}(x) &\triangleq \int_x^\infty f_i(\lambda_j, y) dy \\ &= \begin{cases} (t-s)! \lambda_j^{(s-t)/2} e^{\lambda_j} 2^{(2i-s-t)/2} Q_{s+t-2i+1, t-s}(\sqrt{2\lambda_j}, \sqrt{2x}) & j = 1, \dots, L \\ (t-s)! \Gamma(t+s-i-j+1, x) / (t-j)! & j = L+1, \dots, s \end{cases} \end{aligned} \quad (60)$$

Since $\Pr(\phi_s \geq 0) = 1$, c_1 can also be expressed as

$$c_1 = 1 / |\mathcal{I}_{i,j}(0)|. \quad (61)$$

Substituting (60) and (61) into (57) and simplifying by removing common factors from the numerator and denominator determinants, we obtain the desired c.d.f. of the smallest eigenvalue.

APPENDIX II

PROOF OF THEOREM 2

We can evaluate the c.d.f. of the maximum eigenvalue ϕ_1 as follows

$$F_{\phi_1}(x) = \Pr(\phi_s < \dots < \phi_1 \leq x) = \int_{\mathcal{D}_2} f(\phi_1, \dots, \phi_s) d\phi_1 \cdots d\phi_s \quad (62)$$

where $\mathcal{D}_2 = \{\phi_s < \dots < \phi_1 < x\}$. We evaluate these integrals by following a similar procedure to that used for evaluating (55) in the proof of Theorem 1, to obtain

$$F_{\phi_1}(x) = c_2 \left| \int_0^x f_i(\lambda_j, y) dy \right|, \quad (63)$$

where $f_i(\lambda_j, y)$ is defined as in (58). We obtain the desired result by noting that $c_1 = c_2$ since $\Pr(\phi_1 < \infty) = 1$, and using the property

$$\int_0^x f_i(\lambda_j, y) dy = \int_0^{+\infty} f_i(\lambda_j, y) dy - \int_x^{+\infty} f_i(\lambda_j, y) dy. \quad (64)$$

along with (60) and (61) in (63).

APPENDIX III

PROOF OF THEOREM 3

Let $D_3 = \{\phi_s < \dots < x < \phi_{k-1} < \dots < \phi_1\}$, $D_4 = \{x < \phi_{k-1} < \dots < \phi_1 < +\infty\}$ and $D_5 = \{0 < \phi_s < \dots < \phi_k < x\}$. Using (46) and (56) we can write the probability p as

$$p = c_1 \int_{D_3} \sum_{\sigma} \sum_{\mu} \text{sgn}(\mu) \prod_{k=1}^s \phi_{\sigma_k}^{t-k} e^{-\phi_{\sigma_k}} \{\Upsilon\}_{\sigma_k, \mu_k} d\phi_k. \quad (65)$$

Note that the summation over σ can be decomposed as [12]

$$\sum_{\sigma} = \sum_1 \sum_{r_{\alpha_1}} \sum_{r_{\alpha_2}} \quad (66)$$

where $\sum_{r_{\alpha_1}}$ denotes summation over the permutations $(r_{\alpha_1}, \dots, r_{\alpha_{k-1}})$ of $(1, \dots, k-1)$ and $\sum_{r_{\alpha_2}}$ denotes summation over the permutations $(r_{\alpha_k}, \dots, r_{\alpha_s})$ of (k, \dots, s) . Therefore, we have

$$\begin{aligned} p &= c_1 \sum_1 \sum_{\mu} \int_{D_3} \text{sgn}(\mu) \sum_{r_{\alpha_1}} \sum_{r_{\alpha_2}} \prod_{k=1}^s \phi_{\sigma_k}^{t-k} e^{-\phi_{\sigma_k}} \{\Upsilon\}_{\sigma_k, \mu_k} d\phi_k \\ &= c_1 \sum_1 \sum_{\mu} \text{sgn}(\mu) I_1(\alpha) I_2(\alpha), \end{aligned} \quad (67)$$

where

$$I_1(\alpha) = \sum_{r_{\alpha_1}} \int_{D_4} \prod_{i=1}^{k-1} \phi_{r_{\alpha_i}}^{t-\alpha_i} e^{-\phi_{r_{\alpha_i}}} \{\Upsilon\}_{r_{\alpha_i}, j} d\phi_{r_{\alpha_i}} = \prod_{i=1}^{k-1} \int_x^{+\infty} \phi_{r_{\alpha_i}}^{t-\alpha_i} e^{-\phi_{r_{\alpha_i}}} \{\Upsilon\}_{r_{\alpha_i}, j} d\phi_{r_{\alpha_i}}, \quad (68)$$

$$I_2(\alpha) = \sum_{r_{\alpha_2}} \int_{D_5} \prod_{i=k}^s \phi_{r_{\alpha_i}}^{t-\alpha_i} e^{-\phi_{r_{\alpha_i}}} \{\Upsilon\}_{r_{\alpha_i}, j} d\phi_{r_{\alpha_i}} = \prod_{i=k}^s \int_0^x \phi_{r_{\alpha_i}}^{t-\alpha_i} e^{-\phi_{r_{\alpha_i}}} \{\Upsilon\}_{r_{\alpha_i}, j} d\phi_{r_{\alpha_i}}. \quad (69)$$

The last equality follows from [12, Lemma 1]. The desired result follows from (64), (60) and the definition of the determinant.

APPENDIX IV

PROOF OF THEOREM 4

Here we will derive the p.d.f. expansion (27). The corresponding c.d.f. expansion (28) then follows trivially.

We start by noting that since $f_{\phi_k}(\phi_k) = dF_{\phi_k}(\phi_k)/d\phi_k$, the Taylor expansion of $f_{\phi_k}(\phi_k)$ around the origin can be written as

$$f_{\phi_k}(\phi_k) = F_{\phi_k}^{(1)}(0) + F_{\phi_k}^{(2)}(0)\phi_k + \cdots + \frac{F_{\phi_k}^{(q+1)}(0)}{q!}\phi_k^q + o(\phi_k^q). \quad (70)$$

In order to simplify the derivations, we will initially work with $F_{\phi_k}(x)$ under the assumption of full-rank Ω (i.e. $L = s$). We will then generalize our result to the arbitrary-rank Ω case by evaluating limits where necessary.

A. Derivation for ϕ_1

We first derive the first order expansion of $f_{\phi_1}(\phi_1)$. Using (19) and a well-known result for the k th derivative of a determinant, we obtain

$$F_{\phi_1}^{(q+1)}(x) = \frac{1}{|\Psi(0)|} \sum_{q_1+\cdots+q_s=q+1} \frac{(q+1)!}{q_1!q_2!\cdots q_s!} \left| \frac{d^{q_i}}{dx^{q_i}} \{\Xi(x)\}_{i,j} \right|. \quad (71)$$

We require the q_i th order derivative of the Nuttall Q -function $Q_{s+t-2i+1,t-s}(\sqrt{2\lambda_j}, \sqrt{2x})$ in (20). Omitting details, with the help of Leibnitz' rule, we evaluate these derivatives as follows

$$Q_{s+t-2i+1,t-s}^{(q_i)}(\sqrt{2\lambda_j}, \sqrt{2x}) = - \sum_{r=0}^{q_i-1} \binom{q_i-1}{r} g_1^{(q_i-1-r)}(x) g_2^{(r)}(x) \quad (72)$$

for $q_i \geq 1$, where

$$g_1(x) = \exp(-\lambda_j - x) \quad (73)$$

$$g_2(x) = \left(\sqrt{\lambda_j}\right)^{t-s} \left(\sqrt{2}\right)^{s+t-2i} \sum_{p=0}^{\infty} \frac{\lambda_j^p x^{t-i+p}}{p!(t-s+p)!}. \quad (74)$$

Hence,

$$Q_{s+t-2i+1,t-s}^{(q_i)}(\sqrt{2\lambda_j}, \sqrt{2x}) \Big|_{x=0} = \begin{cases} 0 & q_i - 1 < t - i \\ - \left(\sqrt{\lambda_j}\right)^{t-s} \left(\sqrt{2}\right)^{s+t-2i} \sum_{r=t-i}^{q_i-1} \binom{q_i-1}{r} \frac{e^{-\lambda_j} (-1)^{q_i-1-r} \lambda_j^{r-t+i} r!}{(r-t+i)!(r-s+i)!} & q_i - 1 \geq t - i \end{cases} \quad (75)$$

To obtain the first order expansion of $f_{\phi_1}(\phi_1)$, we require the minimum exponent q in (70) (and corresponding values of q_1, \dots, q_s in (71)), such that $F_{\phi_1}^{(q+1)}(0) \neq 0$. Using (71), (75) and the properties of the determinant, we find that

$$q_i - 1 = t - i + p_i, \quad (76)$$

where $i = 1, \dots, s$ and (p_1, \dots, p_s) is a permutation of $(0, \dots, s-1)$. Thus

$$d_1 = q = \sum_{i=1}^s q_i - 1 = st - 1, \quad (77)$$

and

$$a_1 = \frac{st e^{-\text{tr}(\Lambda)} 2^{s(t-1)/2} \prod_{i=1}^s \lambda_i^{(t-s)/2} \prod_{i < j}^s (\lambda_i - \lambda_j) |\Delta_1|}{|\Psi(0)| \prod_{i=0}^{s-1} (i! (t-s+i)!)}, \quad (78)$$

where

$$\{\Delta_1\}_{i,j} = 1/(t+s-i-j+1). \quad (79)$$

We now simplify a_1 . Using [26, Eq. 13] we can write

$$|\Psi(0)| = 2^{s(t-1)/2} \left(\prod_{i=0}^{s-1} i! \right) \left(\prod_{i=1}^s \lambda_i^{(t-s)/2} \right) \left| L_{s-i}^{(t-s)}(-\lambda_j) \right|. \quad (80)$$

We also manipulate $|\Delta_1|$ by subtracting the first row from all other rows, removing factors, then subtracting the first column from all other column, and again removing factors. This yields

$$|\Delta_1| = \frac{((s-1)!(t-1)!)^2}{(t+s-1)!(t+s-2)!} |\Delta_1^{[1,1]}| \quad (81)$$

where $\Delta_1^{[1,1]}$ is the principle submatrix of Δ_1 , with first row and column removed. Continuing this same process $s-1$ more times, we obtain

$$|\Delta_1| = \frac{(\prod_{i=1}^s (s-i)!(t-i)!)^2}{\prod_{i=1}^{2s} (t+s-i)!} = \frac{(\Gamma_s(s)\Gamma_s(t))^2 \Gamma_{t-s}(t-s)}{\Gamma_{t+s}(t+s)}. \quad (82)$$

Substituting (80) and (82) into (78) and simplifying yields

$$a_1 = \frac{st e^{-\text{tr}(\Lambda)}}{\Gamma_s(t+s)} \frac{\prod_{i < j}^s (\lambda_i - \lambda_j)}{\left| L_{s-i}^{(t-s)}(-\lambda_j) \right|} = \frac{st e^{-\text{tr}(\Lambda)}}{\Gamma_s(t+s)} \frac{|\lambda_j^{i-1}|}{|L_{i-1}^{(t-s)}(-\lambda_j)|}. \quad (83)$$

Finally, we remove the remaining determinant ratio. To do this, we manipulate the numerator determinant in such a way as to construct a scaled version of the denominator determinant. Start by considering the row $i = s$. Using (34), the elements in this row of the denominator determinant

can be written as

$$L_{s-1}^{(t-s)}(-\lambda_j) = \sum_{k=0}^{s-2} \binom{t-1}{i-1-k} \frac{\lambda_j^k}{k!} + \frac{\lambda_j^{(s-1)}}{(s-1)!}, \quad j = 1, \dots, s \quad (84)$$

We construct this row as the last row of the numerator determinant by first dividing the s th row in the numerator determinant by $(s-1)!$, and multiplying the determinant by $(s-1)!$ to compensate. This gives elements corresponding to the $(s-1)^{\text{th}}$ order polynomial terms on the right-hand side of (84). All of the remaining terms on the right-hand side of (84) can be generated from weighted sums of the rows $i = 1, \dots, s-1$ in the numerator determinant (these row operations do not change the value of the determinant). If we then apply the same process to rows $i = s-1, \dots, 1$, in order, at each stage constructing another row of the denominator determinant, and pulling out a multiplicative factor of $(i-1)!$, we obtain

$$\frac{|\lambda_j^{i-1}|}{|L_{i-1}^{(t-s)}(-\lambda_j)|} = \prod_{i=1}^s (s-i)! = \Gamma_s(s). \quad (85)$$

Substituting (85) into (83) gives the desired result for a_1 . Note that in this case, a_1 is only a function of $\text{tr}(\Omega)$, and is independent of the rank of Ω .

B. Derivation for ϕ_k ($k = 2, \dots, s$)

Now consider the case $k > 1$. Starting with (22) for the case of full-rank Ω , We observe that the minimum exponent q is obtained when $(\alpha_1, \dots, \alpha_s) = (1, \dots, s)$. Applying the same steps as for the ϕ_1 case, we obtain

$$q_i = \begin{cases} 0 & i = 1, \dots, k-1 \\ t - i + e_i + 1 & i = k, \dots, s \end{cases} \quad (86)$$

where (e_k, \dots, e_s) is a permutation of $(0, \dots, s-k)$. Hence

$$d_k = q - 1 = \sum_{i=k}^s q_i - 1 = s_k t_k - 1, \quad 2 \leq k \leq s. \quad (87)$$

Combining (77) and (87) yields (29). Also, we obtain

$$a_k = \frac{s_k t_k c_3 2^{s_k(t_k-1)/2} |\Delta_2| |\Delta_3|}{\Gamma_{s_k}(s_k) \Gamma_{s_k}(t_k)} \quad (88)$$

where

$$\{\Delta_2\}_{i,j} = \begin{cases} Q_{s+t-2i+1,t-s}(\sqrt{2\lambda_j}, 0) & i = 1, \dots, k-1 \\ (\sqrt{\lambda_j})^{t+s-2i} e^{-\lambda_j} & i = k, \dots, s \end{cases} \quad (89)$$

and

$$\{\Delta_3\}_{i,j} = 1/(t_k + s_k - i - j + 1) \quad i, j = 1, \dots, s - k + 1. \quad (90)$$

Now, using [26, Eq. 13], we factorize $|\Delta_2|$ as follows

$$|\Delta_2| = \left(\prod_{j=1}^s \lambda_j^{(t-s)/2} \right) \Gamma_{k-1}(s) 2^{(-k^2+k(t+s+1)-(t+s))/2} |\Delta_4| \quad (91)$$

where

$$\{\Delta_4\}_{i,j} = \begin{cases} \sum_{\ell=0}^{s-i} \binom{t-i}{s-i-\ell} \lambda_j^\ell / \ell! & i = 1, \dots, k-1 \\ \lambda_j^{s-i} e^{-\lambda_j} & i = k, \dots, s \end{cases} \quad (92)$$

We evaluate $|\Delta_3|$ using (79) and (82) as

$$|\Delta_3| = \frac{(\Gamma_{s_k}(s_k) \Gamma_{s_k}(t_k))^2 \Gamma_{t_k-s_k}(t_k - s_k)}{\Gamma_{t_k+s_k}(t_k + s_k)}. \quad (93)$$

Now substituting (93) and (91) into (88) and simplifying yields

$$a_k = \frac{s_k t_k \Gamma_{k-1}(s) \Gamma_{s_k}(s_k)}{\Gamma_{s_k}(t_k + s_k)} \frac{|\Delta_4|}{\prod_{i < j}^s (\lambda_i - \lambda_j)} \quad (94)$$

Finally, we generalize this result to arbitrary-rank Ω . To this end, we require the following limit

$$\lim_{\lambda_{L+1} \rightarrow 0, \dots, \lambda_s \rightarrow 0} \frac{|\Delta_4|}{\prod_{i < j}^s (\lambda_i - \lambda_j)}, \quad (95)$$

which we easily evaluate using (53), and substitute into (94) to obtain the final expression.

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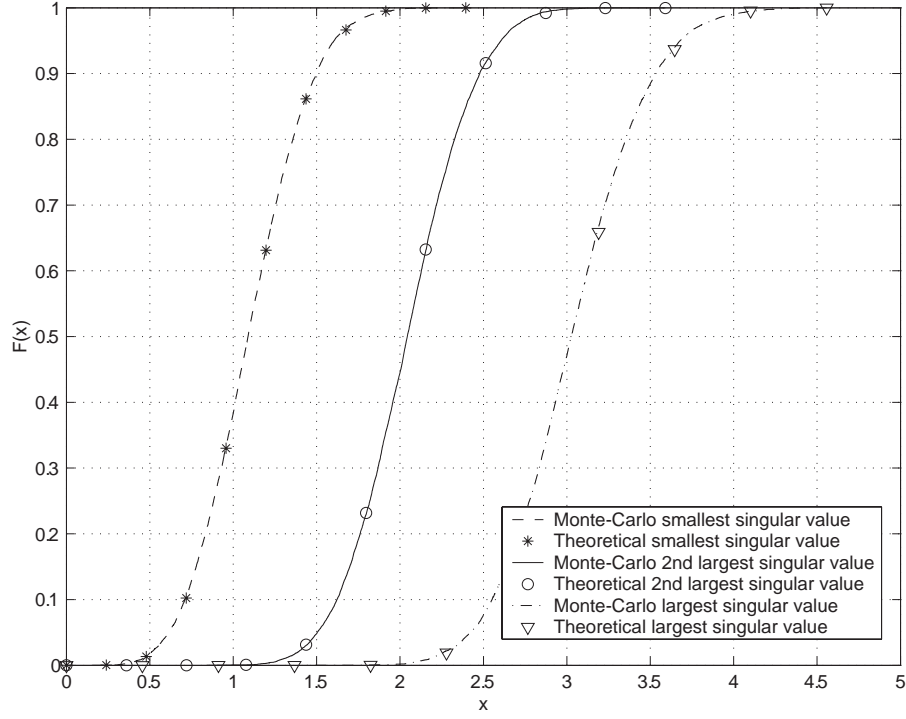


Fig. 1. Ordered c.d.f.s of the singular values of a 3×5 Ricean MIMO channel, with $K = 10\text{dB}$.

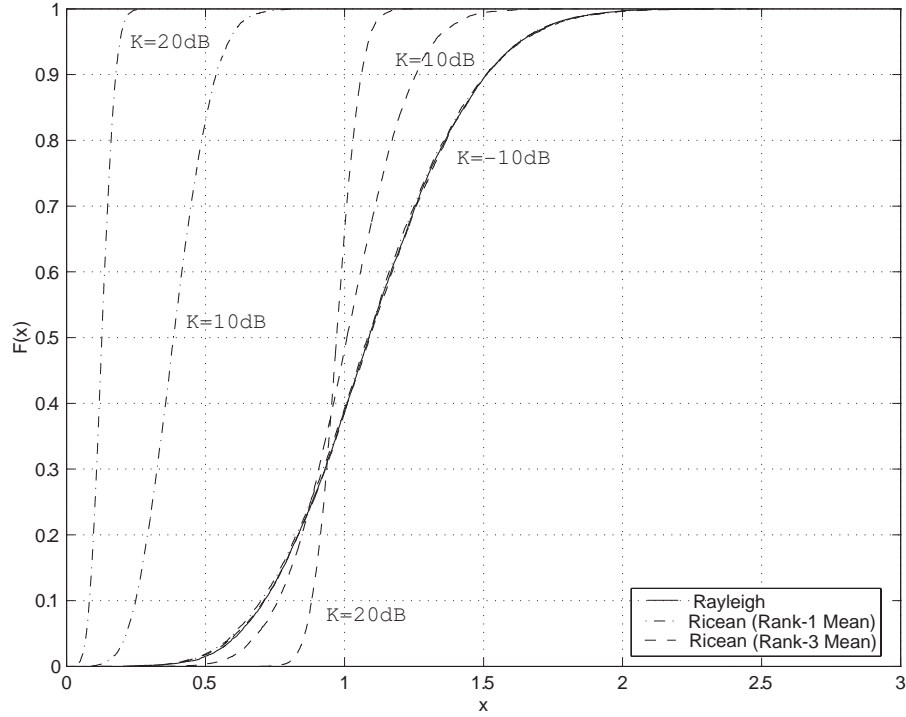


Fig. 2. Smallest singular value c.d.f. for a 3×5 Ricean MIMO channel with rank-1 and rank-3 channel means, and for different K -factors. Rayleigh c.d.f. presented for comparison.

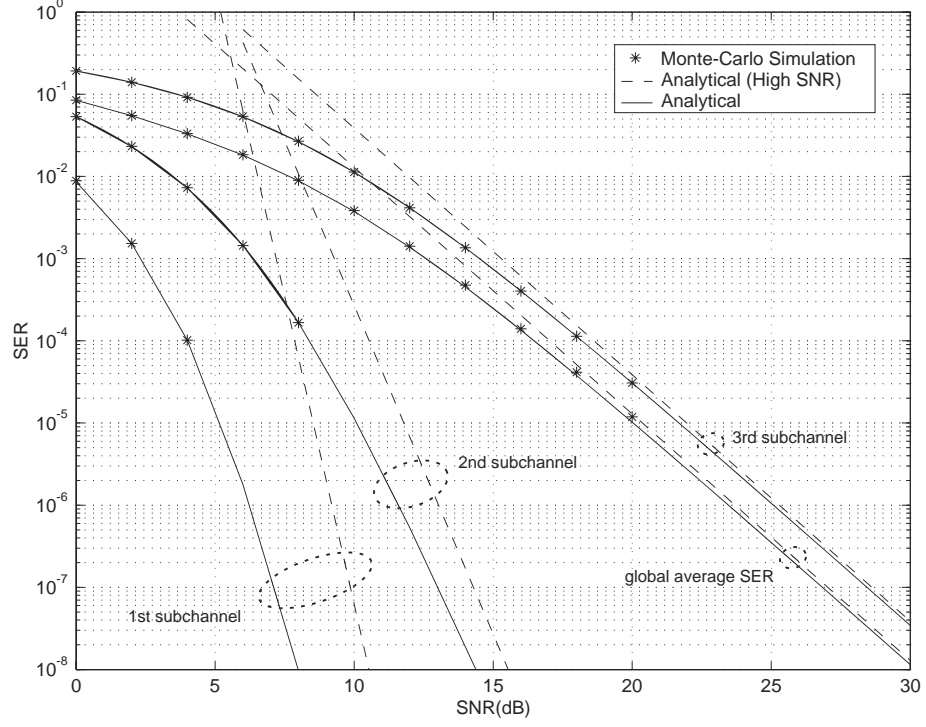


Fig. 3. Exact analytical SER, high SNR analytical SER, and Monte-Carlo simulated SER for 3×5 MB MIMO in uncorrelated Ricean fading, with rank-3 mean matrix and $K = 0$ dB.

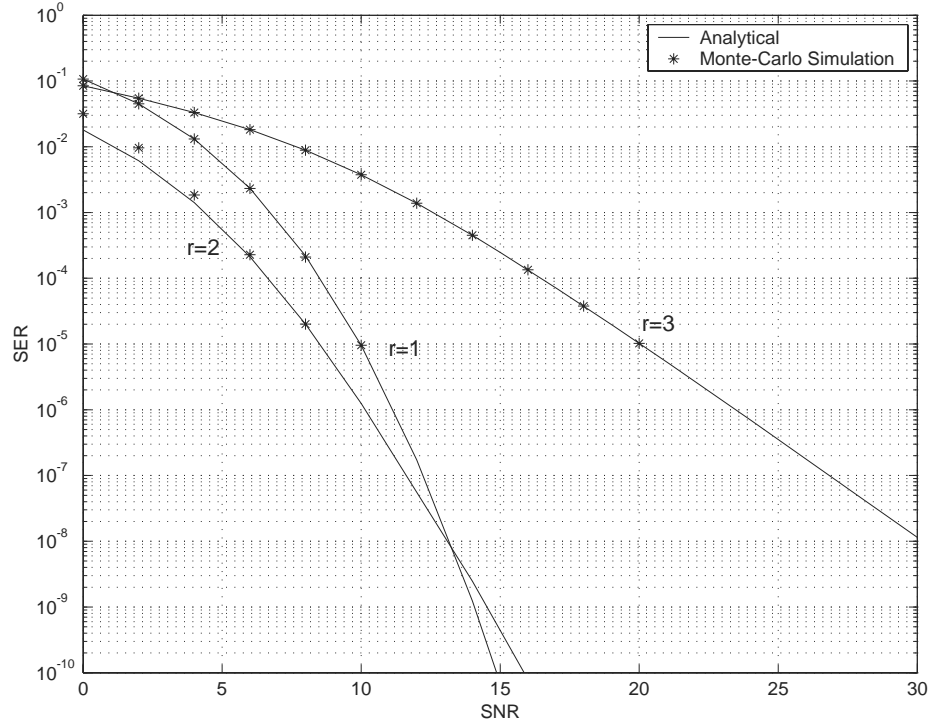


Fig. 4. SER for a 3×5 MB MIMO system with different numbers of active subchannels, in Ricean fading with rank-3 mean matrix and $K = 0$ dB. Spectral efficiency is 3 bits/s/Hz.

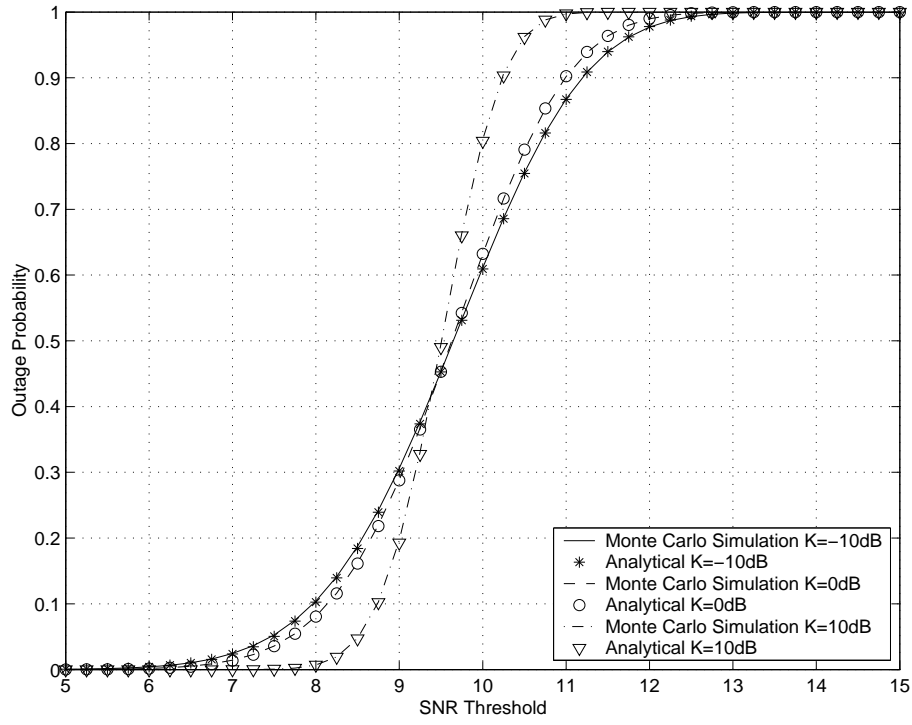


Fig. 5. Outage probability of 3×5 MIMO-MRC in Ricean channels with rank-3 mean matrix, various K -factors, and for $P = 0$ dB.